THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. X.] DECEMBER 1918. [No. 6.

8-J MADI

PROGRESS REPORT.

2. The following gentlemen have been elected members of the Society :—

1. *Mr. Vaman M. Gaitonde, B.A.*, (Hon.)—Lecturer in Mathematics, Wilson College, Chowpatty, Bombay.
2. *Mr. Tenneti Surya Narayana*, B.A., (Cantab)—Lecturer in Mathematics, Ceded Districts College, Anantapur ;
3. *Mr. B. T. Krishna Aiyangar*, B.A.—Mathematics Assistant, Central College, 21 Kottige Lane, Bangalore (City) ;
4. *Mr. Bhimasena Rao*,—Research Student, Mysore University, Central College, Bangalore, (at concessional rate) ;
5. *Mr. M. V. Sundaresan*, B.A.—Teacher, Pachaiyappa's College School, Madras. (Concessional rate);
6. *Mr. S Muthukrishnan*, M.A.—Asst. Lecturer, Government College, Kumbakonam.

2. The recent Influenza Epidemic worked its cruel ravages in our society and we to-day mourn the sad and untimely death of the following members. All the five were young, promising and industrious workers in their own lines of action.

(1) *Mr. R. J. Pocock*, B.A., B.Sc.—Director, Nizamiah Observatory, Begumpet ;

(2) *Mr. J. R. Gunjikar*, B.A., B.Sc.—Assistant Accountant General, Calcutta ;

(3) *Mr. S. V. Setti*, B.A., A.M.I.C.E.—Professor, College of Engineering, Bangalore.

(4) *Mr. D. G. Dandekar* B. Sc., Professor of Physics, Maharajah College, Jaipur.

(5) *The Rev. T. Noronha*, St. Aloysius College, Mangalore.

3. The attention of our members is invited to the leaflet circulated along with the Journal, in connection with the next Conference of our Society.

POONA,
1st *Dec.* 1918.

D. D. KAPADIA,
Honorary Joint Secretary.

Infinite Series and Arithmetical Functions.

BY PROF. F. HALLBERG.

(*Concluded from p.* 264, *Vol. X.*)

IV. Laguerre's Series—(*continued.*)

17. Before proceeding with our analysis of Laguerre's series, it will be well to examine on what foundations our theory rests, in order to distinguish between established and conjectured results.

The following two theorems given by Bromwich (*Infinite Series*, p. 171, ex. 15), are of importance for our purpose.

I, If $v_n(x)$ decreases as n increases and

$$\operatorname*{Lt}_{x\to 1} v_n(x)=1,$$

and if Σa_n is convergent (or divergent to $+\infty$), then

$$\operatorname*{Lt}_{x\to 1} \Sigma a_n v_n(x)=\Sigma a_n \text{ (or} +\infty\text{)}.$$

II. If $A_n=\sum_{h=1}^{n} a_h,\quad B_n=\sum_{h=1}^{n} b_h,$

and if

$$\operatorname*{Lt}_{n\to\infty} B_n/A_n=l,$$

then

$$\operatorname*{Lt}_{x\to 1}\left\{\sum b_n v_n(x)\right\}\Big/\left\{\sum a_n v_n(x)\right\}=l,$$

provided that A_n is always positive, and that A_n tends to ∞.

18. Put now

$$\sum_{n=1}^{\infty} f(n)\frac{x^n}{1-x^n}=\frac{x}{1-x}\sum_{n=1}^{\infty} a_n v_n(x) \quad \dots \quad \dots \quad (18.1)$$

Thus

$$a_n=\frac{f(n)}{n}, \quad \dots \quad \dots \quad \dots \quad (18.11)$$

$$v_n(x)=\frac{nx^{n-1}}{1-x^n}(1-x)$$

$$=\frac{x^{n-1}+x^{n-1}+\dots+x^{n-1}}{1\ \ +x\ \ \ +\dots+x^{n-1}} \quad \dots \quad \dots \quad (18.12)$$

$$\therefore \quad \operatorname*{Lt}_{x\to 1} v_n(x)=1 \quad \dots \quad \dots \quad (18.2)$$

Also

$$v_n(x)-v_{n+1}(x)=$$

$$=\frac{(x^{n-1}-x^n)+(x^{n-1}-x^{n+1})+\dots+(x^{n-1}-x^{2n-1})}{(1+x+\dots+x^{n-1})(1+x+\dots+x^{n-1}+x^n)}>0, \quad \dots \qquad (18.21)$$

if $0<x<1$.

$$\therefore \qquad v_n(x)>v_{n+1}(x) \quad \dots \quad \dots \quad \dots \qquad (18.22)$$

Thus the necessary conditions for $v_n(x)$ are satisfied, and we may apply Bromwich's theorems to Laguerre's series.

We have thus justified to some extent the use to which the series has already been put in the calculation of asymptotic mean values of arithmetical functions. By far the most difficult question, however—whether the asymptotic mean value really *exists*—cannot be solved in this way. But we have a powerful instrument for its calculation, provided its existence is *established* by other means, or *assumed*.

In general, our result, obtained by the aid of Cesaro's theorem, may be written:

$$\underset{n\to\infty}{\mathrm{Lt}}\ \frac{1}{n}\sum_{h=1}^{n}\psi(h)=\sum_{n=1}^{\infty}\frac{f(n)}{(n)} \quad \dots \quad \dots \qquad (18.3)$$

or

$$\frac{1}{n}\sum_{h=1}^{n}\psi(h)\sim\sum_{n=1}^{\infty}\frac{f(n)}{n}, \quad \dots \quad \dots \qquad (18.4)$$

provided the limit exists (or tends to ∞) and the series converges (or diverges to ∞).

We will now consider some additional examples.

19. Using the notation of (11·21), we have

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^{s-1}}\cdot\frac{x^n}{1-x^n}=\sum_{n=1}^{\infty}x^n.\sum_{d:n}\frac{f(d)}{d^{s-1}} \quad \dots \quad \dots \qquad (19{\cdot}1)$$

Thus (18·4) gives

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{f(d)}{d^{s-1}}\sim\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\mathrm{F}(s), \quad \dots \quad \dots \qquad (19{\cdot}11)$$

provided the *asymptotic mean value* (which will in the following be denoted A. M. V.) on the left exists and the Dirichlet's series converges.

Similarly

$$\sum_{n=1}^{\infty} \mu(n) \frac{f(n)}{n^{s-1}} \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} x^n . \sum_{d:n} \mu(d) \frac{f(d)}{d^{s-1}}, \quad \dots \quad (19{\cdot}2)$$

from which follows by (11·22), $f(n)$, satisfying the condition (11·2)

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d;h} \mu(d) \frac{f(d)}{d^{s-1}} \infty \sum_{n=1}^{\infty} \mu(n) \frac{f(n)}{n^s} = \frac{1}{F(s)}, \quad \dots \quad (19{\cdot}21)$$

provided the A. M. V. exists and the Dirichlet's series converges.

The formulae (19·11), (19·21) may be regarded as generalisations of **(13·24)**.—An interesting problem would be to examine, whether the A. M. V. and the series have the same convergence abscissa. This may be considered as very probable.

Using certain well-known formulae from the theory of Dirichlet's series*, we have thus in all probability :

For $f(n) = t(n)$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d:h} \frac{t(d)}{d^{s-1}} \infty \zeta^2(s). \quad \dots \quad \dots \quad \dots \quad (19{\cdot}12)$$

For $f(n) = \int(n)$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d:h} \frac{\int(d)}{d^{s-1}} \infty \zeta(s).\ \zeta(s-1). \quad \dots \quad \dots \quad (19{\cdot}13)$$

For $f(n) = \phi(n)$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d:h} \frac{\phi(d)}{d^{s-1}} \infty \frac{\zeta(s-1)}{\zeta(s)}, \quad \dots \quad \dots \quad (19{\cdot}14)$$

For $f(n) = (-1)^{n-1}$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d:h} \frac{(-1)^{d-1}}{d^{s-1}} \infty (1-2^{1-s}).\ \zeta(s). \quad \dots \quad (19{\cdot}15)$$

For $f(n) = 1$:

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{d:h} \frac{1}{d^{s-1}} \infty \zeta(s), \quad \dots \quad \dots \quad \dots \quad (19{\cdot}16)$$

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* See "Encyclopedie" T. I, Vol. 3, p. 268.

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{\mu(d)}{d^{s-1}}\infty\frac{1}{\zeta(s)}. \quad \dots \quad \dots \quad (19\cdot22)$$

Of these, (19·12), (19·16), (19·22) are probably valid for R $(s)>1$, (19·13), (19·14) for R $(s)>2$, (19·15) for R $(s)>0$.——Formula (19·16) is nothing else than (13·24).

20. Let $f(z)$ be any analytic function, and thus

$$f(z)-f(0)=\frac{z}{1}\cdot f'(0)+\frac{z^2}{2!}f''(0)+ \quad \dots \quad \dots \quad (20\cdot1)$$

Then

$$\sum_{n=1}^{\infty}\frac{z^n}{(n-1)!}f^{(n)}(0).\frac{x^n}{1-x}=\sum_{n=1}^{\infty}x^n.\sum_{d:n}\frac{z^d}{(d-1)!}f^{(d)}(0), \quad \dots \quad (20\cdot11)$$

from which follows

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{z^d}{(d-1)!}f^{(d)}(0)\infty f(z)-f(0), \quad \dots \quad (20\cdot12)$$

for a sufficiently small $|z|$, provided the A. M. V. exists.

In this way we obtain, for instance

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}z^d\infty -\log(1-z), \quad \dots \quad \dots \quad (20\cdot13)$$

when $|z|<1$.

For $z=-1$, the last formula becomes

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}(-1)^{d-1}\infty\log 2 \quad \dots \quad \dots \quad (20\cdot131)$$

Many more examples could of course be given, but would give us nothing fundamentally new.

It is further evident, that any other expansion of a function in the form of an infinite series will furnish similar results. To go into details would carry us to far. The following examples must suffice.

Formula (7.31), for instance, gives

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\frac{2dx^2}{x^2-d^2\pi^2}\infty x\cot x-1, \quad \dots \quad \dots \quad (20\cdot2)$$

and the corresponding (7·33)

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}\mu(d)\left[x\cot\frac{d}{x}-d\right]\infty\frac{2x^2}{x^2-\pi^2}, \quad \ldots \quad (20{\cdot}21)$$

provided the limits on the left exist.

Similarly formula (11·31) gives

$$\frac{1}{n}\sum_{h=1}^{n}\sum_{d:h}d\left(\frac{s}{d}-\frac{s}{s+d}\right)\backsim s(\psi(s)+C)+1, \quad \ldots \quad (20{\cdot}3)$$

when s is not a negative integer, provided the A. M. V. exists; and (11.33) gives a corresponding formula.

21. Let p be a given prime number.

Define $f_p(n)=1$, if $n=p^r$, $r=$positive integer.

$f_p(n)=0$, if n contains other prime factors than p.

$f_p(1)=0$.

$\psi_p(n)=r$, if $n=p^r n'$, $n'\not\equiv 0 \pmod p$.

$\psi_p(n)=0$, if $n\not\equiv 0 \pmod p$.

Thus

$$\sum_{r=1}^{\infty}\frac{x^{p^r}}{1-x^{p^r}}=\sum_{n=1}^{\infty}\psi_p(n)\, x^n. \quad \ldots \quad \ldots \quad (21.1)$$

From this we obtain in the usual manner

$$\frac{1}{n}\sum_{h=1}^{n}\psi_p(h)\backsim\sum_{r=1}^{\infty}p^{-r}=\frac{1}{p-1}. \quad \ldots \quad \ldots \quad (21.11)$$

This result is capable of the following generalisation.

Let $(P)=(p_{n_1}, p_{n_2}, \ldots, p_{n_k})$ be a given finite set of primes, k in number.

Define $f_p(n)=1$, if n is made up entirely of primes, belonging to the given set.

$f_p(n)=0$, if n contains other prime factors than those belonging to the given set.

$f_p(1)=0$.

and $\psi_p(n)=\prod_{i=1}^{k}(1+r_i)-1$, if $n=n'.\prod_{i=1}^{k}p_{n_i}^{r_i}$

$n'\not\equiv 0 (\text{mod } p_{n_i};\ i=1, 2, \ldots, k)$.

Thus

$$\sum_{n=1}^{\infty} f_p(n). \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} \psi_p(n).x^n, \qquad \dots \qquad \dots \qquad (21.2)$$

from which follows

$$\frac{1}{n}\sum_{h=1}^{n} \psi_p(h) \backsim \sum \frac{1}{\pi_p - 1} \qquad \dots \qquad \dots \qquad \dots \qquad (21.21)$$

where the sum to the right is extended over all numbers π_p, made up entirely of primes belonging to (P), and not forming an rth power of any factor of $\prod_{i=1}^{k} p_{n_i}$, r being ≥ 2.

22. Comparing (12·1), (12.11), (12·12) with the corresponding

$$\sum_{n=1}^{\infty} f_1(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} x^n \psi_1(n), \qquad \dots \qquad \dots \qquad (22.1)$$

where

$$\psi_1(n) = \sum_{d:n} f_1(d), \qquad \dots \qquad \dots \qquad \dots \qquad (22.11)$$

$$f_1(n) = \sum_{d:n} \mu(d) \psi_1\left(\frac{n}{d}\right), \qquad \dots \qquad \dots \qquad (22\ 12)$$

and where $f_1(n)$, and hence by (22.11) also $\psi_1(n)$, are *positive*, we obtain easily from Bromwich's theorem II and Cesaro's theorem of comparison for two divergent power series *, the following interesting result, provided

$\sum_{n=1}^{\infty} \frac{f_1(n)}{n}$ *diverges*, and *both limits exist*:

$$\underset{n\to\infty}{\text{Lt}} \frac{\sum_{h=1}^{n} \frac{f(h)}{h}}{\sum_{h=1}^{n} \frac{f_1(h)}{h}} = \underset{n\to\infty}{\text{Lt}} \frac{\sum_{h=1}^{n} \psi(h)}{\sum_{h=1}^{n} \psi_1(h)} \qquad \dots \qquad \dots \qquad (22.2)$$

(For the proof, use the value for $v_n(x)$ given in (18.12)).

* See Bromwich, *Infinite Series*, p. 131,

Ex 1. Let

$$\left.\begin{array}{l} f(n)=\mu^2(n) \\ f_1(n)=1 \end{array}\right\} \quad \therefore \left.\begin{array}{l} \psi(n)=p(n) \\ \psi_1(n)=t(n) \end{array}\right\}$$

Thus

$$\frac{\sum_{h=1}^{n} \frac{\mu^2(h)}{h}}{\sum_{h=1}^{n} \frac{1}{h}} \infty \frac{\sum_{h=1}^{n} p(h)}{\sum_{h=1}^{n} t(h)} \infty \frac{6}{\pi^2}$$

by (13.41), (13.12).

Hence

$$\sum_{h=1}^{n} \frac{\mu^2(h)}{h} \infty \frac{6}{\pi^2} . \log n. \qquad \dots \qquad \dots \qquad (22.21)$$

Ex. 2. Let

$$\left\{\begin{array}{l} f(n)=\sum_{d:n} \mu(d) \cdot \phi\left(\frac{n}{d}\right) \\ f_1(n)=n \end{array}\right\} \left\{\therefore \begin{array}{l} \psi(n)=\phi(n) \\ \psi_1(n)=\int(n) \end{array}\right\}$$

Thus

$$\frac{\sum_{h=1}^{n} \frac{1}{h} \cdot \sum_{d:h} \mu(d) \cdot \phi\left(\frac{h}{d}\right)}{\sum_{h=1}^{n} \frac{1}{h} \cdot h} \infty \frac{\sum_{h=1}^{n} \phi(h)}{\sum_{h=1}^{n} \int(h)} \sim \frac{36}{\pi^4}$$

by (13.32), (13.22)

Hence

$$\sum_{h=1}^{n} \frac{1}{h} \cdot \sum_{d:h} \mu(d) \, \phi\left(\frac{h}{d}\right) \infty \frac{36}{\pi^4} \cdot n. \qquad \dots \qquad (22.22.)$$

Ex. 3. Let

$$\left.\begin{array}{l} f(n)=\phi(n) \\ f_1(n)=n \end{array}\right\} \quad \therefore \quad \left.\begin{array}{l} \psi(n)=n \\ \psi_1(n)=\int(n) \end{array}\right\}$$

Thus

$$\frac{\sum_{h=1}^{n}\frac{\phi(h)}{h}}{\sum_{h=1}^{n}\frac{h}{h}} \sim \frac{\sum_{h=1}^{n} h}{\sum_{h=1}^{n}\int(h)} \sim \frac{6}{\pi^2},$$

by (13.22).

Hence

$$\sum_{h=1}^{n}\frac{\phi(h)}{h} \backsim \frac{6}{\pi^2}.\ n. \quad \dots \quad \dots \quad \dots (22\ 23)$$

Putting $\psi(n)=\frac{\phi(n)}{n}$ in formula (12.1), we obtain from (18.3) by the aid of (22.23)

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{d:n}\mu\left(\frac{n}{d}\right)\frac{\phi(d)}{d} =$$

$$=\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{d:n}\mu\ (d).\ d,\ \phi\left(\frac{n}{d}\right)=\frac{6}{\pi^2}. \quad \dots (22.231)$$

We know, however, from (11.12), that

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}=\frac{1}{\zeta(2)}=\frac{6}{\pi^2},$$

from which we conclude

$$\sum_{d:n}\mu(d).\ d.\ \phi\left(\frac{n}{d}\right)=\mu(n), \quad \dots \quad \dots (22.3)$$

a formula, which is easily verified.

Formula (22.3) may be written

$$\frac{\mu(n)}{n}=\sum_{d:n}\mu(d)\,\frac{\phi\left(\frac{n}{d}\right)}{\frac{n}{d}},$$

from which follows, by (12.12)

$$\frac{\phi(n)}{n}=\sum_{d:n}\frac{\mu(d)}{d}. \quad \ldots \quad \ldots \quad \ldots \ (22.31)$$

Various other general results similar to (22.2) may be obtained by using Frobenius's theorem with Bromwich's extension (See Bromwich, *Infinite Series*, pp. 132, 313).

23. μ-*inversion of Laguerre's series.*

Let in (12.1), $f(n)$ satisfy the condition (11.2), and μ-invert.

The result is

$$\sum_{r=1}^{\infty}\mu(r).\,f(r).\sum_{n=1}^{\infty}x^{nr}.\,\psi(n)=\frac{x}{1-x}. \quad \ldots \quad \ldots \ (23.1)$$

Hence:

$$\sum_{d:n}\mu(d).\,f(d).\,\psi\left(\frac{n}{d}\right)=1, \quad \ldots \quad \ldots \ (23.11)$$

provided always

$$f(m\,m')=f(m)f(m'). \quad \ldots \quad \ldots \quad \ldots \ (23.111)$$

Let now $\psi(n)$ satisfy (11.2), and μ—invert.

We obtain

$$\sum_{r=1}^{\infty}\mu(r).\,\psi(r).\sum_{n=1}^{\infty}\frac{x^{nr}}{1-x^{nr}}.f(n)=x$$

$$=\sum_{n=1}^{\infty}\frac{\mu(n)x^{n}}{1-x^{n}}\text{, by (14.1).} \quad \ldots \quad \ldots \ (23.2)$$

Hence :

$$\sum_{d:n}\mu(d).\,\psi(d).\,f\left(\frac{n}{d}\right)=\mu(n) \quad \ldots \quad \ldots \ (23.21)$$

for $n>1$, *provided always*

$$\psi(mm')=\psi(m)\psi(m') \quad \ldots \quad \ldots \ (23\cdot211)$$

We will consider a few simple examples.

Ex. 1. Put

$$f(n)=1. \quad \therefore\ \psi(n)=t(n).$$

Hence by (23·11)

$$\sum_{d:n} \mu(d).\ t\left(\frac{n}{d}\right) = 1 = f(n),$$

which is evidently correct because of (12·12)

Ex. 2. Put

$$f(n) = n. \quad \therefore\ \psi(n) = \int(n)$$

Hence by (23.11).

$$\sum_{d:n} \mu(d).\ d.\ \int\left(\frac{n}{d}\right) = 1\ .$$

The last formula may be written

$$\sum_{d:n} \mu(d).\ \frac{\int\left(\frac{n}{d}\right)}{\left(\frac{n}{d}\right)} = \frac{1}{n},$$

from which follows, by (12·12), (12·11)

$$\frac{f(n)}{n} = \sum_{d:n} \frac{1}{d} = \sum_{d:n} \frac{d}{n} = \frac{1}{n}.\ \sum_{d:n} d,$$

an evident identity.

Ex. 3. Put

$$f(n) = \phi(n). \quad \therefore\ \psi(n) = n.$$

In this case, (23·21) becomes (22·3).

For examples of a more general nature, use for $f(n)$ or $\psi(n)$ the arithmetical functions defined in § 3. (towards the end of p. 175, Vol. IX).

V. Differentiation of Laguerre's Series.

24. The series (12.1) evidently admits of differentiation. The result may be written:

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \cdot \frac{n^2.\ x^{n-1}}{(1+x+x^2+ \ldots +x^{n-1})^2} = \frac{(1-x)^2}{x} \cdot \sum_{n=1}^{\infty} n.\ \psi(n).\ x^n. \quad (24.1)$$

If we here put

$$v'_n(x) = \frac{n^2.\ x^{n-1}(1-x)^2}{(1-x^n)^2} \quad \ldots \quad \ldots \quad \ldots (24.11)$$

we find that the condition

$$\underset{x\to 1}{\text{Lt}}\ v'_n(x)=1, \qquad \dots \qquad \dots \qquad \dots\ (24.12)$$

is satisfied.

Let $0<y<1$.

$$\therefore\quad 1+(1+y)+(1+y+y^2)+\dots+(1+y+y^2+\dots+y^{n-1})>$$
$$>y^{2n}.\ 1+y^{2n-1}.\ (1+y)+y^{2n-2}.\ (1+y+y^2)+\dots$$
$$+y^{n+1}(1+y+y^2+\dots+y^{n-1}).$$

Multiply by $(1-y)>0$.

$$\therefore\quad (1-y)+(1-y^2)+(1-y^3)+\dots+(1-y^n)>$$
$$>y^{n+1}(1-y^n)+y^{n+2}(1-y^{n-1})+\dots+y^{2n}(1-y).$$

$$\therefore\quad n+ny^{2n+1}>y(1+y+y^2+\dots+y^{2n-1})=y.\ \frac{1-y^{2n}}{1-y}.$$

$$\therefore\quad n-ny+ny^{2n+1}-ny^{2n+2}>y-y^{2n+1}.$$

$$\therefore\quad \frac{n}{1-y^{2n}}>\frac{(n+1)y}{1-y^{2(n+1)}}.$$

Square, put $y=\sqrt{x}$, and multiply by $x^{n-1}(1-x)^2$, when we obtain

$$\frac{n^2x^{n-1}(1-x)^2}{(1-x^n)^2}>\frac{(n+1)^2x^n(1-x)^2}{(1-x^{n+1})^2},$$

or

$$v'_n(x)>v'_{n+1}(x) \qquad \dots \qquad \dots \qquad \dots \qquad \dots\ (24.13)$$

for $0<x<1$.

We may therefore apply Bromwich's theorem I, on the left side of (24.1), using Cesaro's theorem on the right, when we obtain

$$\frac{2}{n^2}\sum_{h=1}^{n} h\,\psi(h)\sim\sum_{n=1}^{\infty}\frac{f(n)}{n}. \qquad \dots \qquad \dots \qquad \dots\ (24.2)$$

provided the limit exists (or tends to ∞*), and the series converges (or diverges to* ∞*).*

We may put (24.2) into a slightly different form.

For this purpose, assume $n\,\psi(n)=\theta(n)$;

$$\therefore\quad f(n)=\sum_{d\,:\,n}\mu(d)\,\psi\left(\frac{n}{d}\right)=\sum_{d\,:\,n}\mu\left(\frac{n}{d}\right)\psi(d)=$$

$$=\frac{1}{n}\sum_{d\,:\,n}\mu(d).\ d.\ \theta\left(\frac{n}{d}\right)=\sum_{d\,:\,n}\mu\left(\frac{n}{d}\right)\frac{\theta(d)}{d}.$$

Hence :

$$\frac{2}{n^2}\sum_{h=1}^{n}\theta(h) \backsim \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{d:n}\mu(d).\,d.\,\theta\left(\frac{n}{d}\right) =$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{d:n}\mu\left(\frac{n}{d}\right)\frac{\theta(d)}{d}, \qquad \ldots \qquad \ldots \qquad (24{\cdot}21)$$

valid under similar conditions.

We will now consider some examples.

25. As in § 19., we have, putting $\frac{f(n)}{n^{s-1}}$ instead of $f(n)$ in (24·2)

$$\frac{2}{n^2}\sum_{h=1}^{n} h \sum_{d:h}\frac{f(d)}{d^{s-1}} \backsim F(s), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (25{\cdot}1)$$

and similarly, *if the condition* (11· 2) *is satisfied* :

$$\frac{2}{n^2}\sum_{h=1}^{n} h \sum_{d:h}\mu(d)\frac{f(d)}{d^{s-1}} \backsim \frac{1}{F(s)}, \quad \ldots \quad \ldots \quad \ldots \qquad (25{\cdot}2)$$

always provided that the limits on the left exists and that the Dirichlet's series on the right converges.

Ex. : For $f(n)=1$, formula (25.1) gives

$$\frac{2}{n^2}\sum_{h=1}^{n} h^{2-s} \sum_{d\cdot h} d^{s-1} \backsim \zeta(s), \quad \ldots \quad \ldots \quad \ldots \qquad (25.11)$$

probably valid for $R(s)>1$.

The case $s=2$ is particularly important. We obtain

$$\sum_{h=1}^{\infty}\int(h) \backsim \frac{\pi^2}{12}n^2, \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (25.111)$$

in agreement with Dirichlet's result, given in formula (13.22). Hence the alternative result (see p. 257 of the present paper) is incorrect.

The case $s=1$ shows that $\sum_{h=1}^{n} h.\,t(h)$ is of an order greater than $n.^2$

For $f(n)=1$, formula (25.2) gives

$$\frac{2}{n^2}\sum_{h=1}^{n} h^{2-s}\sum_{d\,:\,h}\mu\left(\frac{h}{d}\right)d^{s-1}\backsim\frac{1}{\zeta(s)}, \qquad \ldots \qquad \ldots \qquad (25.1)$$

which for $s=2$ becomes

$$\sum_{h=1}^{\infty}\phi(h)\backsim\frac{3}{\pi^2}n^2 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (25.21)$$

in agreement with (13.32).

Other special results may easily be obtained in a way similar to that of § 19

The matter of § § 20, 21, may also be easily extended on the present lines.

26. It is further evident that we may, by the aid of Bromwich's theorem II, obtain a result corresponding to (22·2). Under the same conditions as in § 22, we have :

$$\underset{n\to\infty}{\text{Lt}}\frac{\displaystyle\sum_{h=1}^{n}\frac{f(h)}{h}}{\displaystyle\sum_{h=1}^{n}\frac{f_1(h)}{h}}=\underset{n\to\infty}{\text{Lt}}\frac{\displaystyle\sum_{h=1}^{n}h\,\psi(h)}{\displaystyle\sum_{h=1}^{n}h\,\psi_1(h)} \qquad \ldots \qquad \ldots \qquad (26{\cdot}1)$$

From this result, new asymptotic formula may be obtained

We note (see § 22., Ex. 3.) $\dfrac{\displaystyle\sum_{h=1}^{n}h\int(h)}{\displaystyle\sum_{h=1}^{n}h^2}\backsim\dfrac{\pi^2}{6}$,

But

$$\sum_{h=1}^{n}h^2=\tfrac{1}{6}n(n+1)(2n+1)\backsim\frac{n^3}{3}$$

Hence

$$\sum_{h=1}^{n} hf(h) \sim \frac{\pi^2}{18} n^3. \qquad \dots \qquad \dots \qquad \dots \qquad (26{\cdot}2)$$

27. We may apply μ inversion to formula (24.1) written in the form

$$\sum_{n=1}^{\infty} nf(n) \frac{x^n}{(1-x^n)^2} = \sum_{n=1}^{\infty} n\psi(n)x^n, \qquad \dots \qquad \dots \qquad (27.1)$$

but the results are the same as those of § 23.

We will consider specially the formula (14.22) (where the brackets in the denominator of the fraction under the sum should be removed), comparing it with the corresponding

$$\sum_{n=1}^{\infty} \frac{n\phi(n)x^n}{(1-x^n)^2} = \frac{x(1+x)}{(1-x)^3}. \qquad \dots \qquad \dots \qquad (27.2)$$

We obtain

$$\frac{\displaystyle\sum_{n=1}^{\infty} \frac{\phi(n)}{n} v_n(x)}{\displaystyle\sum_{n=1}^{\infty} \frac{\phi(n)}{n} v_n'(x)} = \frac{1}{1+x}, \qquad \dots \qquad \dots \qquad (27.21)$$

and hence

$$\underset{x \to 1}{\mathrm{Lt}} \frac{\displaystyle\sum_{n=1}^{\infty} \frac{\phi(n)}{n} v_n(x)}{\displaystyle\sum_{n=1}^{\infty} \frac{\phi(n)}{n} v_n'(x)} = \frac{1}{2}, \qquad \dots \qquad \dots \qquad (27.3)$$

This is an interesting relation between two divergent series, corresponding terms of which tend to equality. The anomaly is of course due to the fact, that $v_n(x)$ and $v_n'(x)$ approach unity in a different manner.

VI. Integration of Laguerre's Series.

28. Dividing both sides of (12.1) by x and integrating, we obtain

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} . \log (1-x^n) = -\sum_{n=1}^{\infty} \frac{\psi(n)}{n} . x^n, \quad \ldots \quad \ldots (28.1)$$

or

$$\prod_{n=1}^{\infty} (1-x^n)^{\frac{f(n)}{n}} = e^{-\sum_{n=1}^{\infty} \frac{\psi(n)}{n} . x^n} \quad \ldots \quad \ldots (28.11)$$

no constant of integration being required.

From this result we may deduce many interesting identities.

We note :

$$\log \prod_{n=1}^{\infty} (1-x^n) = -\sum_{n=1}^{\infty} \frac{f(n)}{n} x^n. \quad \ldots \quad \ldots (28.2)$$

$$\log \prod_{n=1}^{\infty} (1-x^n)^{\frac{1}{n}} = -\sum_{n=1}^{\infty} \frac{t(n)}{n} x^n. \quad \ldots \quad \ldots (28.3)$$

$$\log \prod_{n=1}^{\infty} (1-x^n)^{\frac{\mu(n)}{n}} = -x. \quad \ldots \quad \ldots \quad \ldots (28.4)$$

$$\log \prod_{n=1}^{\infty} (1-x^n)^{\frac{\phi(n)}{n}} = -\frac{x}{1-x}. \quad \ldots \quad \ldots \quad \ldots (28.5)$$

29. Formula (28.1) may be written

$$\frac{\sum_{n=1} \frac{\psi(n)}{n} . x^n}{\sum_{n=1}^{\infty} \frac{1}{n} . x^n}$$

$$= \sum_{n=1}^{\infty} \frac{f(n)}{n} + \frac{\sum_{n=1}^{\infty} \frac{f(n) \log n}{n} . \frac{\log (1+x+\ldots+x^{n-1})}{\log n}}{\log (1-x)} \quad \ldots (29.1)$$

Put

$$v''_n(x) = \frac{\log(1+x+\ldots+x^{n-1})}{\log n} \quad \ldots \quad \ldots (29.11)$$

$$\therefore \qquad \operatorname*{Lt}_{x\to 1} v''_n(x) = 1. \quad \ldots \quad \ldots (29.111)$$

Also

$$v''_n(x) > v''_{n+1}(x), \quad \ldots \quad \ldots (29.112)$$

since the fraction $v''_n(x)$ is <1 for $x<1$, and the numerator is increased by a smaller quantity than the denominator, when $v''_n(x)$ is changed into $v''_{n+1}(x)$.

Hence Bromwich's theorem I, may be applied to the series in the numerator of the last term in (29.1), so that, in particular this series approaches

$$\sum_{n=1}^{\infty} \frac{f(n)\log n}{n},$$

if the latter series converges.

Hence we derive the following theorem, by the aid of Cesaro's theorem :

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{\psi(n)}{n} \backsim \sum_{n=1}^{\infty} \frac{f(n)}{n}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots (29.2)$$

provided the limit exists, and the series

$$\sum_{n=1}^{\infty} \frac{f(n)\log n}{n}$$

is convergent.

(This is of course only a particular case of a more general theorem, where the condition for the validity of (29.2) is that the last term in (29.1) should approach zero, as x tends to unity).

30. As an example of the last theorem, substitute in (29.2) $\frac{f(n)}{n^{s-1}}$ for $f(n)$

We deduce, as in §§ 19., 25.,

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h} \sum_{d:h} \frac{f(d)}{d^{s-1}} \backsim F(s), \quad \ldots \quad \ldots (30\cdot 1)$$

provided the limit on the left exists and the Dirichlet's series converges,

The last condition is obtained from the theorem * that if $F(s)$ converges for $s=s_o$,

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s} \quad \dots \quad \dots \quad (30{\cdot}11)$$

converges for $s>s_o$. Hence, when $R(s)$ is equal to the convergence-abscissa of $F(s)$, a special examination is necessary.

Ex. For $f(n)=1$, formula (30·1) gives

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h^s} \sum_{d\,:\,h}^{n} d^{s-1} \; \zeta(s), \quad \dots \quad \dots \quad (30\ 12)$$

which for $s=2$ becomes

$$\sum_{h=1}^{n} \frac{f(h)}{h} \backsim \frac{\pi^2}{6} \log n \quad \dots \quad \dots \quad \dots \quad (30{\cdot}12)$$

Under conditions similar to the above we obtain in addition

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h} \sum_{d\,:\,h}^{n} \mu(d) \frac{f(d)}{d^{s-1}} \backsim \frac{1}{F(s)}, \quad \dots \quad \dots \quad (30{\cdot}2)$$

provided the condition (11.2) *is satisfied.*

Ex. For $f(n)=1$, formula (30·2) becomes

$$\frac{1}{\log n} \sum_{h=1}^{n} \frac{1}{h^s} \sum_{d\,:\,h}^{n} \mu\left(\frac{h}{d}\right) d^{s-1} \backsim \frac{1}{\zeta(s)} \quad \dots \quad \dots \quad (30{\cdot}21)$$

which for $s=2$ gives

$$\sum_{h=1} \frac{\phi(h)}{h^2} \backsim \frac{6}{\pi^2} \log n \quad \dots \quad \dots \quad \dots \quad (30{\cdot}212)$$

31. From (28·1) we deduce, putting $f(n)=\frac{1}{n}$,

$$-\frac{\pi^2}{6} \log(1-x) - \sum_{n=1}^{\infty} \frac{1}{n} x^n \sum_{d\,:\,n} \frac{1}{d} =$$

* *See* Landau: *Handbuch der Lehre von der Verteilung der Primzahlen*, pp. 108, 109.

$$= \sum_{n=1}^{\infty} \frac{x^n}{n}\left(\frac{\pi^2}{6} - \frac{f(n)}{n}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} \log\,(1+x+\ldots+x^{n-1}). \qquad \ldots \quad \textbf{(31.1)}$$

Hence :

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\pi^2}{6} - \frac{f(n)}{n}\right) = \sum_{n=1}^{\infty} \frac{\log n}{n^2}, \quad \ldots \qquad \ldots \quad \textbf{(31.11)}$$

provided the series on the left converges.—This is a formula very similar to (13.52).

Similarly, for $f(n) = \mu(n)$,

$$-\log\,(1-x) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} - x = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log(1+x\ldots+x^{n-1}). \quad \ldots \; (31.2)$$

But the first term on the left vanishes by (11.13).

Hence

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\,(1+x+\ldots+x^{n-1}) = -x, \quad \ldots \quad \ldots \quad (31.21)$$

from which again follows the result (15.3) for $x \to 1$, the series on the left supposed convergent for $x=1$.

By differentiation of (31.21), or directly from (14.1), we deduce

$$\sum_{n=2}^{\infty} \frac{\mu(n)}{n} . \frac{1+2x+3x^2+\ldots+(n-1).\,x^{n-2}}{1+x+x^2+\ldots+x^{n-2}+x^{n-1}} = -1. \quad \ldots \; (31.22)$$

This series oscillates for $x=1$.

We obtain, however, symbolically

$$\sum_{n=2}^{\infty} \frac{\mu(n)}{n} . \frac{\frac{(n-1).\,n}{2}}{n} \backsim -1,$$

and hence

$$\sum_{n=2}^{\infty} \mu(n) \left(1-\frac{1}{n}\right) \backsim \sum_{n=1}^{\infty} \mu(n) \left(1-\frac{1}{n}\right) \backsim$$

$$\backsim \sum_{n=1}^{\infty} \mu(n) - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \backsim -2,$$

from which follows

$$\sum_{n=1}^{\infty} \mu(n) \backsim -2. \quad \ldots \quad \ldots \quad \ldots \quad (31.3)$$

We may therefore regard it as highly probable, that

$$\underset{x \to 1}{\mathrm{Lt}} \sum_{n=1}^{\infty} \mu(n).\, x^n = -2. \quad \ldots \quad \ldots \quad (31.31)$$

This agrees very well with the theorem of Frobenius. In fact, we find, putting

$$\sigma(n) = \sum_{r=1}^{n} \mu(r), \quad \ldots \quad \ldots \quad \ldots \quad (16.12)$$

for N = 83, 91, 126, 127, 128, 131, 134, 144, etc.

that
$$\frac{1}{N} \sum_{n=1}^{N} \sigma(n) = -2$$

exactly.

VII. Conclusion.

In the present paper I have tried to show, how asymptotic formulæ in the theory of numbers may easily be obtained in a manner, somewhat different from the usual, which rests chiefly on the theory of Dirichlet's series. Incidentally I have derived many other interesting results, several of which appear to be novel. I have dealt mainly with the properties of a class of series, bearing the name of Laguerre, and closely associated with that of Dirichlet. The literature, dealing with the latter class, however, is rapidly attaining enormous proportions while that of the former is very inconsiderable, except in the special case, known as Lambert's series. Laguerre's class of series thus deserves a closer attention than hitherto has been paid to it by workers in the theory of numbers. There is reason to believe that the theory of these series might be brought to at least the same degree of perfection as that of Dirichlet's series at present. Probably new light would thus be thrown on some of the most difficult parts of the subject in particular on the still unproved, highly important, proposition of Riemann about the imaginary roots of the ζ—function. The chief usefulness of the series in question lies in the fact, that they form a connecting link between Dirichlet's series on one hand and ordinary power-series on the other, they being associated in a simple manner with both.

SHORT NOTES.

Note on Triangles inscribed in an Ellipse.*

§ 1. Consider a triangle ABC inscribed in the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

and let the eccentric angles of its corners be α, β, γ respectively.

If (x, y) be the centre of either of the in and ex-circles of the triangle ABC, and p_1, p_2, p_3 the perpendiculars from its vertices on the line $x=X$, we shall obviously have

$$BC.p_1 \pm CA.p_2 \pm AB.p_3 = 0\ .$$

But the equation of a circle having double contact with the ellipse along $x=X$ is,

$$x^2+y^2-2e^2xX+e^2X^2-b^2=0,$$

and therefore, if t_1, t_2, t_3 be the tangents from A,B,C, we have

$$t_1=ep_1\ ;\ t_2=ep_2 \text{ and } t_3=ep_3;$$

and consequently,

$$BC.\ t_1 \pm CA.\ t_2 \pm AB.\ t_3=0\ ;$$

which shows that the circumcircle of ABC touches the circle having double contact with the ellipse along the line $x=X$ parallel to its major axis. Similarly for the circle having double contact along the line $y=Y$.

Hence, if the circle ABC be $(x-h)^2+(y-k)^2=R^2$, the abscissae x_1, x_2, x_3, x_4 of the four incentres of ABC, are given as the roots of

$$[2h(h-e^2X)+2k^2-h^2-k^2+R^2+e^2X^2-b^2]^2=R^2\ \{\ 4(h-e^2X)^2+4k^2\ \}\ .$$

Hence, if p_1' p_2' p_3' and p_4' be the perpendiculars from the four incentres of ABC, upon a chord $x=d$,

$$\begin{aligned} e^4p_1'p_2'p_3'p_4' &= e^4(d-x_1)(d-x_2)(d-x_3(d-x_4) \\ &= (h^2+k^2-2e^2dh+e^2d^2+R^2-b^2)-4R^2\ \{\ (h-e^2d)^2+k^2\ \} \\ &= (\delta^2+R^2-r^2)-4R^2\delta^2 \\ &= [\delta^2-(R-r)^2][\delta^2-(R+r)^2]=t^2t'^2, \end{aligned}$$

where δ is the distance between the circumcentre of ABC and the centre of the circle having double contact along $x=d$; and t, t' the lengths of the common tangents of these circles. From this result, it follows that if the circle round ABC were to touch the circle of double contact, since either t or t' is zero, one at least of the quantities p'_1, p'_2, p'_3, p'_4 is zero, so that the chord of contact with the ellipse must pass through the in-or excentre of the triangle.

* Extracted from a paper read by the author before the Conference of the Indian Mathematical Society, December, 1916.

Again, if $(x-X)^2+(y-Y)^2-r_1^2=0$ be one of these circles the lines through the origin parallel to the lines of intersection of this circle with the original ellipse, are.

$$\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-(x^2+y^2)=0,$$

so that $\lambda=\rho^2$, where ρ is the radiant of the ellipse parallel to the chord of intersection. But the invariant relation between the ellipse and the circle gives

$$\lambda^3\Delta-\lambda^2\Theta+\lambda\Theta'-\Delta'=0$$

so that $\Sigma\rho_1^2=\Theta/\Delta$; $\Sigma\rho_1^2\rho_2^2=\Theta'/\Delta$ and $\rho_1^2\rho_2^2\rho_3^2=\Delta'/\Delta$
and the relation $\Theta^2=4\Delta\Theta'$ becomes $(\Sigma\rho_1)^2-4\Sigma\rho_1^2\rho_2^2=0$

or $$\rho_1\pm\rho_2\pm\rho_3=0$$

i.e. the algebraic sum of the three radiants is zero.

§ 2. The following general results, which follow as corollaries from the preceding, have been kindly suggested by Prof. A. C. L. Wilkinson.

If ABCD is a quadrilateral inscribed in a circle, the sixteen inscribed and escribed centres of the four triangles ABC, ABD, BCD, ACD lie by fours on two of perpendicular lines :

Also, $$r=\frac{1}{2R}\left[(x-h)^2+(y-k)^2-R^2\right],\ \text{for any incircle.}$$

Hence

$$r_1+r_2=\frac{1}{2R}\left[(x_1-h)^2+(y_1-k)^2-R^2+(x_2-h)^2+(y_2-k)^2-R^2\right]$$
$$=\frac{1}{2R}\left[(x_1-h)^2+(y_2-k)^2-R^2+(x_2-h)^2+(y_1-k)^2-R^2\right]$$
$$=r_3+r_4\,;$$

i.e. the sum of the radii of the circles inscribed in ABC and ACD is equal to the sum of the corresponding radii of ABD and BCD. We have also similar theorems for the sets of escribed circles. Again when we make

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(x-X)^2=0\ \text{a circle}$$

we also make $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(x-X)^2+\mu\left[(x-h)^2+(y-k)^2-R^2\right]=0$$

a circle, and the radical axis with

$$(x-h)^2+(y-h)^2-R^2=0$$

is the same for both, so that if it touches the first it also touches the second. Hence the point on the circle ABCD where the circle having double contact with the ellipse touches the given circle is the same for all conics through ABCD. Therefore in the special case of the line-pair (as conic) we obtain the following theorem :

If $S_1, S_2 \ldots S_8$ are the points of contact of the 8 circles touching two given straight lines AC, BD and a given circle, then the same 8 points occur also as the points of contact of circles touching AB, CD and the given circles, as well as of the circles touching BC, AD and the given circle.

A. N. Raghavachar.

On poristic Polygons.

The following result (due to Salmon) furnishes a simple method of determining the condition for the porism of a polygon in terms of the invariants of its in-and circum-conics :—

If a triangle is inscribed in a conic $S=0$ whose sides touch respectively the conics

$$S'+lS=0,\ S'+mS=0 \text{ and } S'+nS=0,$$

then

$$\{\odot'-\Delta(mn+nl+lm)\}^2=4[\Delta'+lmn\Delta][\odot+\Delta(l+m+n)]$$

where $\odot$, $\odot'$, Δ, Δ' *are the invariants of the conics* S and S'.

(*Conic Sections*, p. 343.)

For, consider a polygon $A_1A_2 \ldots A_n$ inscribed in a conic $S=0$, and such that all its sides, except A_1A_n, touch a conic S'. Then it is seen that A_1A_n touches a conic of the form $S'+nS=0$, by a repeated application of Salmon's theorem thus: put $l=m=0$, and $n=l_1$; then the side A_1A_3 of the triangle $A_1A_2A_3$ touches a conic $S'+l_1S=0$; again put $l=l_1$, $m=0$, $n=l_2$, then the side A_1A_4 of the $\triangle$ $A_1A_3A_4$ touches $S'+l_2S=0$; and so on. Now since any side A_pA_q may be lookod upon as the open side of a polygon, it follow that A_pA_q touches a conic whose equation is of the form $S'+l_{p,q}\,S=0$, so that if we apply Salmon's result to the two triangles $A_1A_nA_{n-1}$ and $A_1A_{n-1}A_{n-2}$, we obtain

$$[\odot'-\Delta l_{1,n-2}\, l_{1,n}]^2=4\Delta'[\odot+\Delta(l_{1,n-1}+l_{1,n})]$$

and

$$[\odot'-\Delta l_{1,n-2}\, l_{1,n-1}]^2=4\Delta'[\odot+\Delta(l_{1,n-2}+l_{1,n-1})].$$

On simplification these give the result

$$l_{1,n}=\frac{2\odot}{\Delta}\frac{1}{l_{1,n-1}}+\frac{4\Delta'}{\Delta}\frac{1}{l^2_{1,n-2}}-l_{1,n-2}$$

which may be looked upon as a recurring relation from which any $l_{1,n}$ may be determined if the two preceding ones are known.

Obviously the condition that is necessary and sufficient for the polygon to be poristic is that $l_{1,\,n}=0$.

Putting successively $n=3$, 4 in the above recurring formula, we get

$$4\triangle\triangle'\, l_{1,3}=\odot'^2-4\triangle'\odot,$$

and $$(\odot'^2-4\triangle'\odot)^2\, l_{1,4}=8\triangle'\,(\odot'^3-4\triangle\odot'\odot+8\triangle'^2\triangle)$$

which give the well-known results for the case of the triangle and the quadrilateral.

This general method is obviously laborious for higher values of n, but there may exist shorter methods for special cases.

In the case of a pentagon, it is obvious that A_1A_3, A_1A_4 may be taken to touch the same conic $S'+l_{1,3}\,S=0$, while of course, A_3A_4 touches $S=0$. Directly applying salmon's theorem, we obtain

$$[\odot'-\triangle l_{1,3}^2]^2=4\triangle'\,[\odot+2\triangle l_{1,3}]$$

which gives on reduction

$$(\odot'^2-4\triangle'\odot)^3=32\triangle'^2\triangle\,[\Theta'^3-4\triangle'\odot'\odot+8\triangle'^2\triangle].$$

Generally when n is of the form $2m+1$, it will be seen that $l_{1,\,m}$ $l_{1,\,m+1}$.

In the case of a hexagon, similarly it is obvious that the sides of the triangle $A_1A_3A_5$ all touch the same conic $S'+l_{1,3}\,S=0$. Hence Salmon's theorem directly gives

$$[\odot'-3\triangle l_{1,3}]^2=4\,[\triangle'+\triangle l_{13}^3]\,[\odot+3\triangle l_{1,3}]$$

On substituting the value of $l_{1,3}$ this will be seen to be equivalent to

$$3\,[\odot'^2-4\triangle'\odot]^3+16\triangle'\odot[\odot'^2-4\triangle'\odot]^2+96\triangle'^2\triangle\odot'(\odot'^2-4\triangle'\odot)$$
$$+512\triangle'^4\triangle^2=0.$$

In the solution to Q. 831 (Vol. V. J.I.M.S.), Prof. Wilkinson has obtained the condition for the porism of a hexagon in this form. Generally, when n is of the form $3p$, Salmon's theorem directly gives the condition for porism in the form

$$[\odot'-3\triangle l_{1,\,p+1}^2]^2=4[\triangle'+\triangle l_{1,\,p+1}^3][\odot+3\triangle l_{1,\,p+1}].$$

F. H. V. Gulasekharam.

A Geometrical Problem.

Given the circumcircle and the orthocentre of a triangle to find the locus of the in- and ex-centres.

1. Let S, H denote the circumcentre and the orthocentre, and I the *in* or *ex*-centre. Then

$$SI^2=2R.NI, \qquad \dots \qquad \dots \quad (1)$$

where N is the mid-point of SH or *nine points centre*, and R the *given* circumradius. Thus the locus of I is defined by equation (1), and is readily seen to be a *Cartesian*.

For, the above equation may be written

$$(x^2+y^2)^2=4R^2\{(x-c)^2+y^2\} \quad \dots \quad \dots \quad \dots \quad (2)$$

S being the origin of cartesian coordinates, and N the point $(c, 0)$.

Equation (2) may again be put into the form

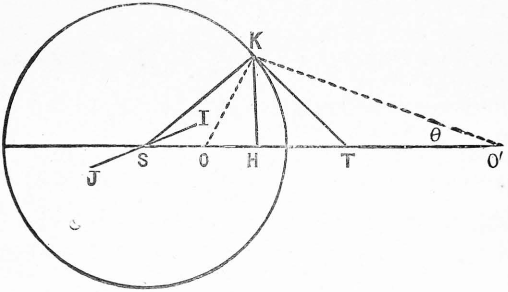
$$\{(x^2+y^2-2R^2\}^2+4R^2(2cx-R^2-c^2)=0,$$

or

$$S^2+I^2u=0, \dots \quad \dots \quad \dots \quad \dots \quad (3)$$

where s is a circle, u a straight line and I the line at infinity. The form (3) shows that the locus is a cartesian, whose double tangent is $u=0$; the points of contact being the real points of intersection of the curve and the circle s.

2. The focal properties of the curve are found as follows :—



From T the inverse of H with respect to the circumcircle draw TK tangent to the circle. Measure TO, TO′ equal to TK. Then O, O′ are two foci of the Cartesian.

For, the circle OKO′ touches SK at K, and therefore SO/SK $=$OK/O′K$=\cot$ KO O′$=m$ (say)

∴ $$SO=mR.$$

and $$SO'=R/m.$$

Also $$SH.ST=SK^2,$$

∴ $$cR\left(m+\frac{1}{m}\right)=R^2.$$

∴ $$c=Rm/(1+m^2).$$

Denoting IO, IO′ by ρ, ρ', we have

$$\rho^2=(x-Rm)^2+y^2;$$
$$\rho'^2=(x-R/m)^2+y^2.$$
$$\therefore \quad \rho^2-m^2\rho'^2=(x^2+y^2-R^2)(1-m^2),$$
$$\rho^2+m^2\rho'^2=(x^2+y^2+R^2)(1+m^2)-4xmR.$$
$$\therefore \quad \{\rho^2-m^2\rho'^2\}^2-2k^2\{\rho^2+m^2\rho'^2\}$$
$$=(x^2+y^2-R^2)^2(1-m^2)^2-2k^2\{(x^2+y^2+R^2)(1+m^2)-4xmR\}$$
$$=(1-m^2)^2(x^2+y^2)^2-2(x^2+y^2)\{R^2(1-m^2)^2+k^2(1+m^2)\}$$
$$+8xk^2mR+R^4(1-m^2)^2-2k^2R^2(1+m^2)$$
$$=k^4, \text{ after reduction by using (2),}$$

where $k^2=R^2(1-m^2)^2/(1+m^2)=(SO'-SO)^2/(1+1/m^2)=OK^2$.

Hence, the relation between ρ and ρ' is

$$\{\rho^2-m^2\rho'^2\}^2-2k^2(\rho^2+m^2\rho'^2)+k^4=0,$$

which can be reduced to the form

$$\rho\pm m\rho'\pm k=0,$$

shewing that the foci are O, O′.

3. Now $\rho+m\rho'>m(\rho+\rho')$, since $m<1$ in the figure; and $(\rho+\rho')>OO'>KO'$ or k/m.

Hence $\rho+m\rho'>k$, for real values of ρ, ρ'.

The *conjugate ovals* are therefore given by $\rho-m\rho'\pm k=0$, the upper and lower signs corresponding to the inner and outer ovals.

4. The equations of the ovals in question may be written in the form

$$\rho-\rho'.\tan\theta\pm 2a\sin\theta=0;$$

or

$$\rho\cos\theta-\rho'\sin\theta\pm a\sin 2\theta=0;$$

if $2a$ is written for OO′.

The fundamental property of this oval may therefore be stated as follows:—

If I is a point on the inner oval, produce IS to J such that JS = SI, and with J as centre and a radius equal to 2R describe a circle cutting the outer oval in I_1, I_2, I_3. Then the system I I_1 I_2 I_3 will be orthocentric, and the harmonic triangle ABC which is inscribed in the circle S will have the fixed point H for its orthocentre.

M. T. Naraniengar.

Astronomical Notes.

In Memorium.

During the last two years the Notes under this head were being written by Mr. R.J. Pocock, B.A., B.Sc, F.R.A.S., late Director of the Nizamiah Observatory, Hyderabad (Deccan), who fell a victim early last month to that fell disease, the Spanish Influenza. The deceased was much attached to his work at the Observatory, and in 1917 published a "*Star Chart*", which consists of a single circular disc which can be made to rotate so as to show at a glance the stars visible in India for any hour in the year, Very recently in the current year, H. E. H. the Nizam's Government published his "*Astrographic Catalogue*." He wrote frequently in the *Monthly Notices of the Royal Astronomical Society*; and in India he appealed to the popular reader through his charming weekly notes on astronomical topics in the *Madras Times*. The Society regrets the loss of such an enthusiastic member; and this column, in particular, will be poorer for the lack of his contributions.

In his memory this column in the current issue has been left vacant.

SOLUTIONS.

Question 930.

(S. Malhari Rao) :—Fill up the vacant cells in the following figure with the remaining numbers of the series 1, 2, 3 ..63, 64, so that the whole figure and each of the four corner subsquares of 16 cells may all be pandiagonal magic squares. (The figure of the question is the square printed below of which only the left hand bottom corner subsquare of 16 cells was filled up. *Ed*).

Solution by R. J. Pocock, B. M. Vasavada B. Sc. and Sadanand.

[*N. B.*—There is a misprint in the question where the number 24 n the cell (6, 1) should be 34].

We may write the given quarter square in the form

$4\times8+7$	$3\times8\times4$	$4\times8\times6$	$3\times8+1$
$4\times8+2$	$3\times8+5$	$4\times8+3$	$3\times8+8$
$3\times8+3$	$4\times8+8$	$3\times8+2$	$4\times8+5$
$3\times8+6$	$4\times8+1$	$3\times8+7$	$4\times8+4$

We can construct 3 other cells in which the cofactors of 8, viz. 4 and 3 are replaced by 5, 2 ; 6, 1 ; 7, 0 respectively, we have then merely to the numbers 1, 2, 3...8 in the subsquares so that each and the whole will be a magic square. We begin by writing down the numbers 1, 2, 3, 4 arrange respectively in four superposable solutions of the eight queens problem which are consistent with the numbers already given. The remaining squares are then easily filled in, adding we obtain the solution as follows :—

$5\times8+5$	$2\times8+2$	$5\times8+\delta$	$2\times8+3$	$7\times8+1$	6	$7\times8+4$	7
$5\times8+4$	$2\times8+7$	$5\times8+1$	$2\times8+6$	$7\times8+\delta$	3	$7\times8+5$	2
$2\times8+1$	$5\times8+6$	$2\times8+4$	$5\times8+7$	5	$7\times8+2$	8	$7\times8+3$
$a\times8+\delta$	$5\times8+3$	$2\times8+5$	$5\times8+2$	4	$7\times8+7$	1	$7\times8+6$
$4\times8+7$	$3\times8+4$	$4\times8+6$	$3\times8+1$	$6\times8+3$	$8+8$	$6\times8+2$	$8+5$
$4\times8+2$	$3\times8+5$	$4\times8+3$	$3\times8+\delta$	$6\times8+6$	$8+1$	$6\times8+7$	$8+4$
$3\times8+3$	$4\times8+\delta$	$3\times8+2$	$4\times8+5$	$8+7$	$6\times8+4$	$8+6$	$6\times8+1$
$3\times8+6$	$4\times8+1$	$3\times8+7$	$4\times8+4$	$8+2$	$6\times8+5$	$8+3$	$6\times8+8$

Question 945.

(SADANAND) :—Fill up the vacant cells of the following magic square :—(The Square printed below with only the two diagonals filled)

Solution (1) *by B. M. Vasavada. B. Sc.*, (2) *by N. P. Pandya*, (3) *by E. Rama Iyer, and* (4) *by Mehta Kanhyalal.*

(1)

46	47	31	65	16
80	54	8	39	24
5	52	27	37	84
1	50	81	35	38
73	2	58	29	42

(2)

46	24	66	53	16
3	54	64	3	45
20	41	27	48	69
63	50	25	35	32
73	36	23	30	43

(3)

46	57	33	53	16
40	54	30	29	42
36	21	27	41	80
10	50	86	35	24
73	23	29	37	43

(4)

46	44	51	48	16
47	54	37	39	28
31	32	27	49	66
8	50	60	35	52
7	25	30	34	43

Question 947.

(M. K. KEWALRAMANI) :—A triangle ABC inscribed in any ellipse touches a confocal ellipse at the points D,E,F respectively. Show that the ratio of the triangle DEF to ABC is r/2R, where r and R refer to the triangle ABC.

Solution (1) *by K. B. Madhava and Hemraj*, (2) *by A. C. L. Wilkinson.*

(1) It is known that D, E and F are the points of contact of the escribed circles of ABC with its sides : cf. Smith : *Geometrical conics* P. 179 Ex 68. (Hence the actual areals of D, E, F are O, $(s-b)/a, (s-c)/a$; etc. We have therefore

$$\frac{\triangle DEF}{\triangle ABC} = \frac{1}{abc}\begin{vmatrix} o & s-b & s-c \\ s-a & o & s-c \\ s-a & s-b & o \end{vmatrix}$$

$$= 2(s-a)(s-b)(s-c)/(abc)$$

$$= 2.\ \frac{\triangle^2}{s.}\ \frac{1}{4R\triangle} = \frac{r}{2R}.$$

(2) The ellipse in trilinears being $lyz + mzx + nxy = 0$, we have for the tangential equation of any confocal—

$$p^2l^2 + q^2m^2 + r^2n^2 - 2plm - 2qrmn - 2rpnl$$
$$+ \lambda(p^2 + q^2 + r^2 - 2pq \cos C - 2qr \cos A - 2pr \cos B) = 0$$

This touches the sides of the triangle of reference if

$$l^2 = m^2 = n^2 = -\lambda$$

Thus we take $yz + zx + xy = 0$ for the ellipse circumscribing ABC and for the confocal

$$pq \sin^2\frac{C}{2} + qr \sin^2\frac{A}{2} + rp \sin^2\frac{B}{2} = 0.$$

The tangential equation of the point of contact with (1, 0, 0) is

$$q \sin^2\frac{C}{2} + r \sin^2\frac{B}{2} = 0.$$

Whence D is $(0, s-b, s-c)$.

It follows that $AF = CD = s-b$; $BF = CE = s-a$; $AE = BD = s-c$ and the ratio of the areas of the triangles ABC, DDF is at once seen to be $\frac{r}{2R}$.

Question 948.

(M. K. Kewalramani) :—A perfectly elastic particle, acted on by on forces is projected from the centre of a rectangle whose sides are $2a$ and $2b$ $(a > b)$ to strike the bigger side first and then go on rebounding from its sides. If it ever pass through an angular point show that the direction of projection makes with the smaller side an angle whose tangent is $\frac{a}{b}.\frac{2n+1}{2m+1}$ where m and n are integers; and if it ever strike any point of the smaller side other than the extremities, the tangent of the angle must be of the form $\frac{a}{b}.\frac{2n+1}{2m}$

Solution by S R. Ranganathan, M.A., L.T.

Let the particle first impinge on the longer side at a distance of x from its midpoint. Then the angle of projection is $\tan^{-1}\frac{x}{b}$, so that it has to be shown that if the particle ever passes through a corner, x must be of the form $\frac{a(2n+1)}{2m+1}$.

Measuring distances parallel to the longer side from its midpoint and neglecting the *sense* of the distances, it is easily seen, because the particle is perfectly elastic, that between two consecutive impacts on a longer side, the particle is carried through a distance $2x$, irrespective of its impinging on a shorter side in the interval.

Hence, after m impacts at the longer sides, the particle will be at a distance of $(2m+1)x$. If it ever paases through a coner, this distance must be a, or $(2n+1)a$, n being any integer. Hence, we get

$$x = \frac{a(2n+1)}{2m+1}.$$

The second part of the question does not seem to be correct. For taking the simple case of $a = \frac{3}{2}b$ and the angle of projection being $\frac{\pi}{4}$, it is easily sesn, that the second impact of the particle will be with a shorter side at a point other than an extremity. But, in such a case, according to the question $\tan\frac{\pi}{4}$ *i.e.* 1 must be of the form

$$\frac{a}{b}.\frac{2n+1}{2m} \text{ i.e. } \frac{3}{2}.\frac{2n+1}{2m}$$

which is untrue, since $3(2n+1)$ is odd and $4m$ is even.

But the condition that the angle should be of the given form is necessary that the particle may impinge at the midpoint of a shorter side.

Question 949.

(C. KRISHNAMACHARY) :—With the usual notation for the numbers of Bernoulli and Euler, show that

(1) $$2n = B_1 2^2(2^2-1)\binom{2n}{2} - B_2 2^4(2^4-1)\binom{2n}{4} + B_3 2^6(2^6-1)\binom{2n}{6} - \ldots + (-1)^{n-1} B_n 2^{2n}(2^{2n}-1)$$

(2) $$(-1)^n E_n (2n+1) = (-1)^n B_n 2^{2n}(2^{2n}-1\binom{2n+1}{1} + (-1)^{n1} B_{n-1} 2^{2n-2}(2^{2n-2}-1)\binom{2n+1}{3} \ldots + \ldots - B_1 2^2(2^2-1)\binom{2n+1}{2} + (2n+1).$$

Solution by Hemraj.

(1) This is given in the reverse after putting

$$S_{2n-1} = \frac{2^{2n}(2^{2n}-1)}{2n}B_{2n-1}.$$

in Art. 575 (7) of Edwards Diff. Cal.

(2) Following the notation of Arts: 571, 572, 573, we have

$$\sin x = \cos x \left[S_1 \frac{x}{1!} + S_3 \frac{x^3}{!} + S^5 \frac{x^5}{5!} + \ldots\ldots \right]$$

Expanding $\sin x$ and $\cos x$, and equating the coefficients of x^{2n+1}, we get

$$S_{2n+1} = {}^{2n+1}C_2 S_{2n-1} - {}^{2n+1}C_4 S_{2n-3} + {}^{2n+1}C_6 S_{2n-5} - \ldots \ldots + (-1)^n \ldots \qquad \ldots \qquad \ldots \quad \text{(A)}$$

Again $\sec x + \tan x = 1 + S_1 \frac{x}{1!} + S_2 \frac{x^2}{2!} + S_3 \frac{x^3}{3!} + \ldots = \sigma(\text{say})$

Differenting

$$\sec x\,(\sec x + \tan x) = \frac{d\sigma}{dx} \text{ i.e. } \sec x + \tan x = \cos x \frac{d\sigma}{dx}$$

Equating the coefficients of x^{2n}, we have

$$S_{2n} = S_{2n+1} - 2nC_2 S_{2n-1} + \ldots (-1)^n S_1$$
$$= ({}^{2n+1}C_2 - {}^{2n}C_2) S_{2n-1} - ({}^{2n+1}C_4 - {}^{2n}C_4) S_{2n-3} + ({}^{2n+1}C_6 - {}^{2n}C_6) S_{2n-5} - \ldots + (-1)^{n-1}({}^{2n+1}C_{2n} - {}^{2n}C_{2n}) S_1 + (-1)^n \quad \text{from (A)}$$
$$= {}^{2n}C_1 S_{2n-1} - {}^{2n}C_3 S_{2n-3} + {}^{2n}C_5 S_{2n-5} - + (-1)^n$$
$$= 2^{2n}(2^{2n-1}) B_{2n-1} - \frac{2n(2n-1)}{3!} B_{2n-3} 2^{2n-2}(2^{2n-2}-1) + \ldots\ldots + (-1)^{n-1}{}_{2n}B_1 \frac{2^2(2^2-1)}{2} + (-1)^n.$$

$$\therefore\ (-1)^n(2n+1)\, E_n = (-1)_n B^n 2^{2n}(2^{2n}-1)\binom{2n+1}{1} + (-1)^{n-1} B_{n-1} 2^{2n-2}(2^{2n-2}-1)\binom{2n+1}{3}$$

$$+\ldots-B_1 2^2(2^2-1)\frac{(2n+1)}{2}+(2n+1).$$

or
$$\tan x = S_1 x + S_3 \frac{n^3}{3!} + \ldots\ldots$$

Differentiating
$$\sec x = \cos x \left[S_1 + S_3 \frac{x^2}{3!} + \ldots \right]$$

$$i.e.\ 1+S_2\frac{x^2}{2!}+\ldots\ldots = \left[1-\frac{x^2}{2!}+\ldots\right]\left[S_1+S_3\,\delta\frac{x^2}{2!}+\ldots\right]$$

Equating coefficients of x^{2n}, we get the result stated.

Question 950.

(C. KRISHNAMACHARI) ;—Collect the coefficients of x^n in the series

$$\frac{1}{1-x}-\frac{x}{(1-x^2)}+\frac{1.3}{2!}\frac{x^2}{(1-x)^3}-\frac{1.3.5}{3!}\frac{x^3}{(1-x)^4}+\ldots$$

and show that their sum is zero when n is odd, and is

$$(-1)^m.\frac{1.3.5\ldots(2m-1)}{2.4.6\quad(2m)}$$

when n is $2m$.

Solution by Hemraj, Sadanand, Alpha, B. M. Vasavada and S. R. Ranganathan.

The given series io equal to

$$\frac{1}{1-x}\left\{1-\tfrac{1}{2}\left(\frac{2x}{1-x}\right)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}\left(\frac{2x}{1-x}\right)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\left(\frac{2x}{1-x}\right)^3+\ldots\right\}$$

$$=\frac{1}{1-x}\left\{1+\frac{2x}{1-x}\right\}^{-\frac{1}{2}}$$

$$=\frac{1}{1-x}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}$$

$$=(1-x^2)^{-\frac{1}{2}}$$

$$=1+\frac{1}{2}x^2+\frac{1.3}{2,4}x^4+\frac{1.3.5}{2.4.6}x^6+\ldots+\frac{1.3.5.-\ldots(2m-1)}{2.4.6.\quad(2m)}x^{2m}+\ldots$$

The absence of an odd power of x clearly shows that the sum of the coefficient of x^n is zero when n is odd.

The coefficient of x^{2m} is $\frac{1.3\ldots.(2m-1)}{2.4\ldots.\ (2m)}$, which shows that the sum is $\frac{1.3.5\ldots.(2m-1)}{2.4.6\ldots.\ (2m)}$ when n is equal to $2m$. The factor $(-1)^m$ with which it is multiplied in the question as printed is incorrect.

Question 951

(C. Krishnamachary):—Prove that

$$(1) \quad \int_0^{\frac{\pi}{4}} (\log \tan x)^{2n} dx = \frac{E_n}{2^{2n+2}} \pi^{2n+1},$$

$$(2) \quad \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos x (\cos x + \sin x)} dx = -\frac{(2^{2n-1}-1)}{2n} B_n \pi^{2n},$$

$$(3) \quad \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos 2x} dx = -\frac{(2^{2n}-1)}{4n} B_n \pi^{2n},$$

$$(4) \quad \int_0^{\frac{\pi}{4}} \frac{(\log \tan x)^{2n-1}}{\cos x \ (\cos x - \sin x)} dx = -\frac{2^{2n-1} B_n}{2n} \pi^{2n}.$$

Solution (1) *by A. C. L. Wilkinson* ; (2) *by Hemraj and Sadanand.*

For (4), we have

$$\int_0^1 \frac{(\log x)^{2n-1}}{1-x} dx = \sum_{r=0}^{\infty} \int_0^1 (\log x)^{2n-1} x^r dx$$

$$= -\sum_0^{\infty} \frac{(2n-1)}{(r+1)^{2n}} = -\frac{2^{2n-1} B_n}{2n} \pi^{2n}.$$

(See : Carr : *Synopsis*, It 2395—2397, whence the formula

$$B_n = 4^n \int_0^{\infty} \frac{y^{2n-1}}{e^{2\pi y}-1} dy$$

is derived ; also, Ramanujan : I. M. J., Vol III, *p.* 226.)

For (2), write x^2 in place of x and use the formula

$$\frac{2x}{1-x^2} = \frac{1}{1-x} - \frac{1}{1+x}.$$

For (3), add (2) and (4).

For (1), we repalce it by

$$\int_0^1 \frac{(\log x)^{2n}}{1+x^2} dx = \sum_0^{\infty} (-) \frac{(2n)\,!}{(2r+1)^{n+1}}.$$

In this write $x = e^{-\pi x}$ and deduce the integrals

$$\int_0^\infty \frac{\cosh 2ax}{\cosh \pi x}dx=\frac{1}{2}\sec a,\int_0^\infty \frac{\cos 2ax}{\cosh \pi x}dx=\frac{1}{2}\text{sech } a,\left(-\frac{\pi}{2}\angle a\angle\frac{\pi}{2}\right)$$

(See Boole ; *Finite Differences* p. 110).

[Compare also : Bromwich *Infinite Series*—*pp.* 452 and 473.]

(2) In each case the result follows by putting $\log \tan x=z$ and applying Edwards, *Diffl. Calc.* § 574.

Question 952.

(HEMRAJ):—The polar reciprocal of an equiangular spiral with respect to a rect. hyperbola having its centre at the pole and touching the spiral is the curve itself.

Solution (1) *by Sadanand*, (2) *by Alpha and L. S. Vaidyanathan, Sadanand.*

(1) Let the hyperbola and the spiral be $pr=a^2$, (2) and $r \sin\alpha=p$. Then if (R,β) be any point, its polar with respect to the hyperbola is $r\cos(\theta+\beta)=\frac{a^2}{R}$.

If it touches the spiral, then $\frac{a^2}{R}=p=r\sin\alpha$, giving $Rr=\frac{a^2}{\sin\alpha}$. The symmetry of this result shows that the polar will touch the spiral at (R,β).

Also it is clear that the curves touch where $R=r$ and hence the result.

(2) Let the rectangular hyperbola be $r^2\cos 2\theta=c^2$, and the spiral be

$$r=a\,e^{\theta\cot\alpha}.$$

These two touch at the point whose vectorial angle is $\frac{1}{2}(\pi-\alpha)$, if

$$c^2=a^2.\sin\alpha.e^{\left(\frac{\pi}{2}-\alpha\right)\cot\alpha}, \quad \dots \quad \dots \quad (1)$$

Now, the tangent to the spiral at (r_1, θ_1) is

$$r\cos(\theta-\overline{\theta_1-90+\alpha})=r_1\sin\alpha=a.\sin\alpha.e^{\theta_1\cot\alpha}.$$

i.e. $$r\sin(\alpha+\theta_1-\theta)=a\sin\alpha\, e^{\theta_1.\cot\alpha} \quad \dots \quad \dots \quad (2)$$

If the pole of this with respect to the rect. hyp. be (ρ,ϕ), its equation must be identical with

$$r.\rho.\cos(\theta+\phi)=c^2. \quad \dots \quad \dots \quad (3)$$

Hence equating coeffts of $r\cos\theta$ and $r\sin\theta$ in (2) and (3) we get

$$\frac{\sin(\alpha+\theta_1)}{\rho.\cos\phi}=\frac{\cos.(\alpha+\theta_1)}{\rho.\sin\phi}=\frac{a.e^{\theta_1\cdot\cot\alpha}.\sin\alpha}{c^2};$$

whence $\tan\phi=.\cot(\alpha+\theta_1)$, or $\phi=\frac{\pi}{2}-(\alpha+\theta_1)$;

and
$$\frac{1}{\rho}=\frac{a.e^{\theta_1\cot\alpha}\sin\alpha}{c^2}$$

$$=\frac{a\sin\alpha\, e^{\left(\frac{\pi}{2}-\alpha-\phi\right)\cot\alpha}}{a^2.\sin\alpha\, e^{\left(\frac{\pi}{2}-\alpha\right)\cot\alpha}}=\frac{e^{-\phi\cot\alpha}}{a}$$

or
$$\rho=a\,e^{\phi\cot\alpha}.$$

Hence the polar reciprocal is the locus of (ρ, ϕ) and is the original spiral itself'

Question 953.

(COMMUNICATED BY HEMRAJ):—Two of the common tangents to a circle S and a conic T meet in P and the other two in Q. Show that P and Q lie on the same confocal to T.

Solution by A. C. L. Wilkinson.

Consider the following theorem.

S. S′ are two conics, C is a conic having double contact with S, then a conic can be described through the intersections of S′ and C which will touch a pair of common chords of S and S′ at the points where the chord of contact of the conics S, C meets them.

Let AB, CD be a pair of common chords of S and S′ and take ABC as triangle of reference; so that

$$S \text{ is } yz+zx+xy=0,$$
$$S' \text{ is } ayz+bzx+cxy=0.$$

Then CD is $(a-c)\,y+(b-c)\,x=0.$

Then C is $xy+yz+zx+(\lambda x+\mu y+\nu z)^2=0,$

and consider the conic

$$xy+yz+zx+(\lambda x+\mu y+\nu z)^2-\frac{1}{c}\,(ayz+bzx+cxy)=0,$$

This passes through the intersections of S′ and C and since it can be written

$$z\{(c-a)'y+(c-b)x\}+c(\lambda x+\mu y+\nu z)^2=0,$$

it is a conic touching $z=0$, $(c-a)y+(c-b)x=0$ at the points where these lines are met by $\lambda x+\mu y+\nu z=0$.

The reciprocal theorem is

S. S′ are two conics and C a conic having double contact with S, then a conic can be described touching the four common tangents of C and S′ and passing through two of the opposite vertices of the quadrilateral formed by the four common tangents to S and S′ and the tangents at these points are the lines joining them to the pole of the chord of contact of the conics S and C with respect to S or C.

Let C degenerate to two points on S and then project these points into the circular points at infinity and the theorem as stated follows with the additional theorem that the tangents at P and Q to the confocal pass through the centre of thegiven circle. This is of course immediately obvious geometrically.

It is curious to notice that we can project this theorem into its own converse, for if we project P, Q into the circular points at infinity, the theorem to be proved becomes :

If from two points Ω, Ω′ lying on a conic C tangents be drawn to a confocal S, these tangents will touch a circle.

To prove the theorem in this form, reciprocate with respect to a focus, and we get the following theorem.

If two tangents to a circle meet another circle in four points, these fou. points lie on a conic having one focus at a limiting point of the two circles

This may be proved as follows :—

Take the pair of tangents

$$S\,\Sigma-\Delta\,(lx+my+n)^2=0$$

to the circle $S=0$ and consider conics through their intersection with $S'=0$, then if $S+\lambda S'=0$ be a point circle,

$$(S+\lambda S')\,\Sigma-\Delta\,(lx+my+n)^2=0$$

is a conic fulfilling the required conditions.

QUESTIONS FOR SOLUTION.

998. (F. H. V. GULASEKHARAM) :—If $u_1+u_2+u_3+u_4=2x$;

$$s_r=\text{sn}\,(u_r, k);\; c_r=\text{cn}(u_r, k)_s d_r=\text{dn}\,(u_r, k)\,;$$

$$s'_r = \frac{\partial s_r}{\partial u_r};\; c'_r=\frac{\partial c_r}{\partial u_r};\; d'_r=\frac{\partial d_r}{\partial u_r};$$

prove that $E\,(u_1)+E(u_2)+E(u_3)+E(u_4)-2E\,(x)$

$$=\frac{k^2c_1c_2c_3c_4}{k'^2-k^2c_1c_2c_3c_4}\sum_{r=1}^{4}\frac{c_r'}{c_r}$$

$$+\frac{2k^2\,\text{sn}\,x\,\text{dn}\,x\,\text{cn}(x-u_3-u_2)\,\text{cn}(x-u_2-u_4)\,\text{cn}\,(x-u_4-u_3)}{k'^2-k^2\,\text{cn}\,x\,\text{cn}\,(x-u_3-u_2)\,\text{cn}\,(x-u_2-u_4)\,\text{cn}\,(x-u_4-u_3)}$$

[*Suggested by Q.* 973.]

Prove also that the above expression is equal to the symmetrical result

$$\frac{k^2k'^2\,s_1s_2s_3s_4\displaystyle\sum_{r=1}^{4}\frac{s'_r}{s_r}+k^2c_1c_2c_3c_4\sum_{r=1}^{4}\frac{c'r}{cr}-d_1d_2d_3d_4\sum_{r=1}^{4}\frac{d'r}{dr}}{k'^2k^2k'^2s_1s_2s_3s_4-k^2c_1c_2c_3c_4+d_1d_2d_3d_4.}$$

999. (LAKSHMISHANKAR N. BHATT and K. J. SANJANA) :—Prove that the following function of the four quantities p, q, r, s—

$$2\Sigma p^2q^4+2\Sigma p^3q^2r-2\Sigma p^4qr-4\Sigma p^3q^3-6\Sigma p^2q^2r^2$$

is equal to 1536 $\text{cosec}^4\,4\alpha$, when p, q, r, s stand for the tangents or cotangents of the angles $\alpha,\ \alpha+\frac{\pi}{4},\ \alpha+\frac{2\pi}{4},\ \alpha+\frac{3\pi}{4}$.

Prove further that the same function has the value $8\cos^2 2\alpha$, when p, q, r, s stand for the sines or cosines of the angles

$$\alpha,\ \alpha+\frac{\pi}{2},\ \alpha+\frac{2\pi}{2},\ \alpha+\frac{3\pi}{2}.$$

1000. (K. J. SANJANA) :—Prove that

$$(n-1)^{-4}+(n+1)^{-4}+(3n-1)^{-4}+(3n+1)^{-4}+(5n-1)^{-4}+(5n+1)^{-4}+\dots$$
$$=\frac{\pi^4}{48n^4}\sec^2y\,(3\sec^2y-2),$$

$$(n-1)^{-5}-(n+1)^{-5}+(3n-1)^{-5}-(3n+1)^{-5}+(5n-1)^{-5}-(5n+1)^{-5}+\dots$$
$$=\frac{\pi^5}{96n^5}\tan y\sec^2y\,(3\sec^2y-1),-$$

where $y\equiv\pi/2n$ and n is an integer greater than unity.

1001. (K. J. Sanjana, M.A.) :—Prove that

$$1+(2n-1)^{-4}+(2n+1)^{-4}+(4n-1)^{-4}+(4n+1)^{-4}+\ldots$$
$$=\frac{\pi^4}{48n^4}\operatorname{cosec}^2 y\,(\operatorname{cosec}^2 y-2).$$

$$1+(2n-1)^{-6}+(2n+1)^{-6}+(4n-1)^{-6}+\ldots\ldots$$
$$\frac{\pi^6}{960n^6}\operatorname{cosec}^2 y\,(15\operatorname{cosec}^4 y-15\operatorname{cosec}^2 y+2),$$

$$1+(2n-1)^{-8}+(2n+1)^{-8}+\ldots$$
$$=\frac{\pi^8\operatorname{cosec}^2 y}{80640n^8}(315\operatorname{cosec}^6 y-420\operatorname{cosec}^4 y+126\operatorname{cosec}^2 y-4),-$$

where $y\equiv\pi/2n$ and n is any positive integer.

1002. (N.P. Pandya) :—Find the lowest prime numbers which can be arranged into a magic square of nine cells. Also find a magic square of nine cells, all numbers in it being perfect squares.

1003. (S. R. Ranganathan) :—If $\phi_n(x)$ stands for the *nth.* Bernoulli's polynomial, show that

$$\phi_n\left(\frac{1}{4}\right)=2^{n-1}\left[\phi_n\left(\frac{1}{8}\right)-\phi_n\left(\frac{3}{8}\right)\right],$$

or
$$=\frac{2^{3n-2}+2^{2n-2}-2^{n-1}}{2^{3n-1}+2^{n-1}-1}\left[\phi_n\left(\frac{1}{8}\right)\;+\;\left(\frac{3}{8}\right)\right],$$

according as n, which is greater than 1, is odd or even.

1004. (C. Krishnamachari) :—Show that the effects of the elliptic motion of the earth on aberration in latitude and longitude are given respectively by

$$-C\sin\beta\,[\sin(\odot-\lambda)-e\sin(\varpi-\lambda)]$$

and
$$-C\sec\beta\,[-\cos(\odot-\lambda)+e\cos(\varpi-\lambda)]$$

where $2ra=U.T.C.\sqrt{1-e^2}\sin 1''$ the notation being as in Ball's *Spherical Astronomy.*

1005. (K. B. Madhava) :—Show that when $|x|<1$, and the real part of z is positive and less than 1, $\Gamma(1-z)=\lim_{x\to 1}(1-x)^{1-z} f(x, z)$, where

$$f(x, z)=\sum_{n=1}^{\infty}\cdot\frac{x^n}{n^z}.$$

1006. (K. B. Madhava) :—It is known that the hypergeometric series $F(a, b, c, x)$ diverges if $a+b=c$. Show that as $x\to 1-0$, the series however, becomes infinite like

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\log\left(\frac{1}{1-x}\right).$$

1007. (Alpha) :—Prove that if

$$S_0=1\,!-2\,!+3\,!-4\,!+\ldots,$$

then $\quad 1(1\,!)-2\,(2!)+3\,(3\,!)-4\,(4\,!)+\ldots=1-2\,S_o,$
$1^2(1\,!)-2^2\,(2\,!)+3^2\,(3\,!)\;-4^2\,(4\,!)+\ldots=5\,S_o-2\,;$
and generally $\Sigma(-)^{n-1}\,n^k\,(n\,!)$ is of the form $\alpha S_o+\beta$, where α and β are integers (positive or negative).

(Bromwich : *Infinite Series*, p. 482, Ex. 21)

1008. (Communicated by Mr. Martyn M. Thomas) :—If I_n is written for the integral $\int_0^{\frac{\pi}{2}} \cos 2\,nx \log\,(2\cos\tfrac{1}{2}\,x)\,dx$,

prove that $\quad \frac{1}{n}\,I_{n+1}-\frac{1}{n+1}\,I_n = \frac{(-)^{n+1}}{2n(n+1)(2n+1)}\,;$

and deduce that

$$2\,n\,I_n=\left(1-\frac{1}{3}+\frac{1}{5}-\ldots=\frac{1}{2n-1}\right)-\frac{\pi}{4}.$$

1009. (Communicated by Mr. Hemraj) :—A B C is a triangle ; O, P are its circumcentre and orthocentre. If O P meets AB, AC in B_1, C_1 ; prove that the join of the circumcentre and the orthocentre of AB_1C_1 is parallel to BC.

1010: (K. J. Sanjana, M.A.,) :—Solve the equation $x^3+y^3=a^3+b^3$ where a and b are given positive quantities and x and y are required to be postive. Examples :—

$$x^3+y^3=19\equiv\left(\frac{5}{2}\right)^3+\left(\frac{3}{2}\right)^3,\quad x^3+y^3=91=4^3+3^3.$$

1011. (Selected):—In an ellipse the tangent at P cuts the directrices in Z, Z′, and the remaining tangents from Z, Z′ to the ellipse meet at T. Shew that P T is normal to the ellipse and bisected by the minor axis.

1012. (N. Sankara Aiyar) :—Solve

(1) $x^6+12x^5+45x^4+100x^3+120x^2+78x+21-0$

(2) $2x^4+12x^3+30x^2+32x+12=0$

by expressing the left hand members in the forms (A^6+B^6) and (A^4+B^4), respectively.

1013. (S. R. Ranganadhan) :—If $N=a_1a_2a_3a_4$ where the a's are mutually prime integers show that the product of the numbers less than N and prime to it is given by

$$\frac{(a_1a_2a_3a_4-1)\,!\;\prod(a_1a_2-1)\,!}{\prod\left\{\,a_1^{(a_2-1)(a_3-1)(a_4-1)}(a_1a_2a_3-1\,!\;\;(a_1-1)\,!\,\right\}}$$

1014. (S. R. Ranganadhan):—Prove this extension of Wilson's Theorem. If m be any nonprime number, then

$\pi d(m)+(1-1)^{\frac{m-1}{2}}\equiv m$ when m iseven $\pi d(m)+(-)^{\frac{m-1}{2}}\equiv m$ when m is odd where $\pi d(m)$ denotes the product of numbers less than m and prime to it.

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